

CONES IN HOMOTOPY PROBABILITY THEORY

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ABSTRACT. This note defines cones in homotopy probability theory and demonstrates that a cone over a space is a reasonable replacement for the space. The homotopy Gaussian distribution in one variable is revisited as a cone on the ordinary Gaussian.

1. INTRODUCTION

This note is concerned with a simple categorical aspect of homotopy probability theory. Ordinary probability theory is concerned with a vector space of random variables together with a linear functional called the expectation. The space of random variables is also endowed with an associative product. The expectation does not respect the associative product. Rather, the failure of the expectation to be an algebra map defines correlations among the random variables.

Homotopy probability theory [1, 2] is a theory in which the space of random variables is a chain complex and the expectation map is a chain map to the ground field. In homotopy probability theory, the chain complex of random variables is also endowed with an associative product. Just as in ordinary probability theory, the expectation map is not assumed to respect the associative product of random variables. The differential is also not assumed to respect the product structure. Homotopy probability theory generalizes ordinary probability theory—every ordinary probability space is a homotopy probability space where the space of random variables is a chain complex concentrated in degree zero with zero differential. The article [2] explains how to obtain meaningful, homotopy invariant correlations among the random variables by adapting the failure of the expectation to be an algebra map, involving the failure of the differential to be a derivation.

In this paper, cones on homotopy probability spaces are introduced. A cone is a factorization of the expectation map into an inclusion followed by a quasi-isomorphism, the result of which is that all of the relations among expectation values become encoded in a differential. A cone on an ordinary probability space is a homotopy probability space that serves as a good replacement for the original probability space and in the cone, homological methods can be used to compute expectation values and correlations of the original probability space. In a variation where the expectation is factored as an inclusion that is an algebra map followed by a quasi-isomorphism, computations in the cone are more simply related to those in the original probability space and can be easier to carry out. One source of these *algebraic* cones is when the relations among expectations arise from a group acting on the random variables.

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2. PRELIMINARIES

In ordinary probability theory where random variables are measurable functions on a measure space and expectation is integration, one has the special random variable 1 which serves a multiplicative identity for the product of random variables. The only compatibility between the algebra structure in the space of random variables and the expectation is the normalization that expectation value of the random variable 1 equals the complex number 1. In order to capture this normalization in homotopy probability theory, pointed chain complexes will be used. It is worth noting that this normalization effectively “cancels” the transcendental unknowns involved in integration.

Definition 2.1. A *pointed chain complex* is a chain complex (V, d) together with a map $v : \mathbb{C} \rightarrow V$ called the *unit*. A morphism of pointed chain complexes is a chain map commuting with the units. The map v can be conveniently identified with the element $v(1) \in V$.

The field of complex numbers \mathbb{C} is considered as a chain complex concentrated in degree zero with zero differential. The identity $\text{id} : \mathbb{C} \rightarrow \mathbb{C}$ makes \mathbb{C} a pointed chain complex.

Definitions 2.2 through 2.8 are taken with slight modification from [2]. The interested reader can find more details there.

Definition 2.2. A *unital commutative homotopy probability space* is a pointed chain complex V together with the following data:

- A graded commutative, associative product on V . The unit $v : \mathbb{C} \rightarrow V$ of the chain complex is the identity for the product. No compatibility is assumed between the product and the differential.
- A map of pointed chain complexes $e : V \rightarrow \mathbb{C}$ called *the expectation*.

A morphism of unital commutative homotopy probability spaces is simply a morphism of pointed chain complexes commuting with expectation. No compatibility is assumed between the morphism and the products. A unital commutative homotopy probability space will be called a *probability space* for short. If the underlying pointed chain complex is concentrated in degree zero and has zero differential, we will call it an *ordinary probability space*.

Note that \mathbb{C} becomes an ordinary probability space by defining the expectation to be the identity map. The assumption that expectation is a map of pointed chain complexes means that $ev = \text{id}_{\mathbb{C}}$. Therefore, both the unit of a probability space $v : \mathbb{C} \rightarrow V$ and the expectation $e : V \rightarrow \mathbb{C}$ are morphisms of probability spaces and the unital conditions on morphisms imply that any map between \mathbb{C} and a probability space must be one of these two. Therefore, \mathbb{C} is both initial and terminal in the category of probability spaces.

In order to define correlations among random variables in a probability space, the failures of (1) the expectation to be an algebra map and (2) the differential to be a derivation must be taken into account. A convenient way to organize the failure to be a derivation is with the language of L_∞ algebras. Most concepts and calculations for an L_∞ algebra V can be expressed in terms of the symmetric (co)algebra of V .

Definition 2.3. Let V be a graded vector space. We denote the cofree nilpotent commutative coalgebra on V by SV ; it is linearly spanned by symmetric powers of V .

Given a symmetric associative product on V , define a map φ from SV to V as follows:

$$a_1 \odot \cdots \odot a_n \mapsto a_1 \cdots a_n.$$

The map φ uniquely extends to be a coalgebra automorphism $SV \rightarrow SV$. By abuse of notation, this coalgebra automorphism will also be denoted φ . In [7, 5], φ is called *the cumulant map*.

Most concepts and calculations for an L_∞ algebra V can be transported by φ .

Definition 2.4. Let V be a graded vector space with an associative product. Let f be a \mathbb{C} -linear map $SV \rightarrow SV$. We call the composition $\varphi^{-1}f\varphi$ the map f *transported by φ* , and denote it f^φ . If V and W are two graded vector spaces with associative products, then a \mathbb{C} -linear map $SV \rightarrow SW$ can also be transported. In this case, $f^\varphi = \varphi^{-1}f\varphi$ where the φ on the right is a coalgebra automorphism $SV \rightarrow SV$ and the φ^{-1} on the left is the inverse of the coalgebra automorphism $\varphi : SW \rightarrow SW$.

Let V be a probability space. The differential d on V can be extended to a square zero coderivation on SV . This makes the probability space into an L_∞ algebra. Transporting this L_∞ structure by φ defines another L_∞ structure d^φ on V called *the transported structure*.

Definition 2.5. Let V be a probability space. A *collection of n homotopy random variables* is an L_∞ morphism from $(\mathbb{C}^{\times n}, 0)$ to (V, d^φ) . That is, it is a degree zero map $X : S\mathbb{C}^{\times n} \rightarrow SV$ satisfying $d^\varphi X = 0$.

Remark 2.6. In [2], a collection of homotopy random variables was defined as the homotopy class of such a morphism, rather than a single morphism. This definition is less obfuscatory.

The expectation e can be viewed as an L_∞ morphism from (V, d) to $(\mathbb{C}, 0)$. This map can be transported by φ .

Definition 2.7. The *total cumulant K* of a probability space is the expectation map transported by φ :

$$K := e^\varphi$$

To summarize: the expectation is a map $e : V \rightarrow \mathbb{C}$ satisfying $ed = 0$. Transporting the expectation map by φ results in the total cumulant, which is a coalgebra map $K : SV \rightarrow S\mathbb{C}$ satisfying $Kd^\varphi = 0$. A coalgebra map $K : SV \rightarrow S\mathbb{C}$ is completely determined by its components, which are multilinear maps $\{k_n : V^{\times n} \rightarrow \mathbb{C}\}$. If V is an ordinary probability space, then these multilinear maps $\{k_n\}$ coincide precisely with the classical cumulants in ordinary probability theory [6].

Definition 2.8. The *joint cumulant* of a collection of n homotopy random variables denotes the composition of the total cumulant map with the collection of n homotopy random variables.

$$(\mathbb{C}^{\times n}, 0) \xrightarrow{X} (V, d) \xrightarrow{K} (\mathbb{C}, 0)$$

Recall that two L_∞ morphisms $X, Y : SV \rightarrow SW$ are homotopic if there is an L_∞ morphism $H : SV \rightarrow SW \otimes \mathbb{C}[t, dt]$ with $H(0) = X$ and $H(1) = Y$.

Proposition 2.9. *If two collections of n homotopy random variables are homotopic then they have identical joint cumulants.*

Proof. See the proof of Lemma 3 in [2]. □

3. CONES AND ALGEBRAIC CONES

3.1. Contractible probability spaces. Here, we identify a condition under which the converse of Proposition 2.9 is true.

Definition 3.1. A probability space is called *contractible* if the expectation map is a quasi-isomorphism.

Proposition 3.2. *Let V be a contractible probability space. Two collections of n homotopy random variables in V are equal if and only if their cumulants are equal.*

Proof. Since V is contractible, the total cumulant is a quasi-isomorphism (since it is the transport of a quasi-isomorphism). Since K is a quasi-isomorphism, two collections of homotopy random variables are homotopic if and only if their compositions with K are homotopic. Since \mathbb{C} has no differential, this is true if and only if their joint cumulants are equal. \square

Definition 3.3. Let V be a probability space. A *cone* on V is a factorization $V \rightarrow CV \rightarrow \mathbb{C}$ of the expectation $V \rightarrow \mathbb{C}$ for which CV is contractible and $V \rightarrow CV$ is an injective map of probability spaces.

One can always factor a chain map $U \rightarrow V$ as an inclusion followed by a surjective quasi-isomorphism $U \hookrightarrow V \xrightarrow{\sim} W$. In particular, the expectation of a probability space $e : V \rightarrow \mathbb{C}$ can be factored in this way, so cones on a probability space always exist.

3.2. Algebra preserving morphisms. While morphisms of probability spaces preserve expectation values, they must be transported in order to relate joint cumulants.

Let $\alpha : V \rightarrow W$ be a morphism of probability spaces. One can relate the joint cumulants by a transported α as follows. Suppose $X : (\mathbb{C}^{\times n}, 0) \rightarrow (V, d^\varphi)$ is a collection of homotopy random variables and $K_V : (V, d^\varphi) \rightarrow (\mathbb{C}, 0)$ and $K_W : (W, d^\varphi) \rightarrow (\mathbb{C}, 0)$ are the total cumulants of V and W respectively, then the joint cumulants satisfy $K_V X = K_W \alpha^\varphi X$. The following diagram illustrates the relationship between α and the joint cumulants.

$$\begin{array}{ccccc}
 & & & K_V & \\
 & & & \curvearrowright & \\
 \mathbb{C}^{\times n} & \xrightarrow{X} & V & \xrightarrow{\alpha^\varphi} & W & \xrightarrow{K_W} & \mathbb{C} \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
 & & V & \xrightarrow{\alpha} & W & \xrightarrow{e_W} & \mathbb{C} \\
 & & & \curvearrowleft & e_V & &
 \end{array}$$

Proposition 3.4. *Morphisms of probability spaces which preserve the algebra structure preserve total cumulants.*

Proof. Let $\alpha : V \rightarrow W$ be a morphism of probability spaces. Then $K_W \alpha^\varphi = K_V$. If the map α preserves the algebra structure then $\alpha^\varphi = \alpha$. So, $K_W \alpha = K_V$. \square

Definition 3.5. Let $V \rightarrow \mathbb{C}$ be a probability space. An *algebraic cone* on V is a cone $V \rightarrow CV \rightarrow \mathbb{C}$ on V for which $V \rightarrow CV$ is an algebra map.

Lemma 3.6. *There exists an algebraic cone on any probability space.*

Proof. Decompose V linearly into $H \oplus B \oplus \hat{B}$, where H is a space of homology representatives including 1, B is the image of d , and \hat{B} is a space of coimages of d so that d is an

isomorphism from \hat{B} to B . Let K be the kernel of the expectation restricted to H ; then H is linearly spanned by K and 1. Define CV to be $V \oplus K[1]$ where $K[1]$ is a shifted copy of K . Extend d by sending $K[1]$ to K by the degree shift. Let the product of $K[1]$ with anything in $K[1] \oplus B \oplus \hat{B} \oplus K$ to be zero. Define the expectation of $K[1]$ to be zero. The evident inclusion from V to CV is a morphism of probability spaces which respects the product structure. \square

Remark 3.7. The fact that there exists an algebraic cone on any probability space allows an ordinary probability space to be replaced by homotopy probability space with nice properties. If $V \rightarrow \mathbb{C}$ is an ordinary probability space, then for any collection of elements $x_1, \dots, x_n \in V$, the map $X : \mathbb{C}^n \rightarrow V$ defined by $e_i \mapsto x_i$ where e_i is the i -th standard basis vector of \mathbb{C}^n defines a collection of homotopy random variables. The cumulant of X is an L_∞ morphism $KX : \mathbb{C}^n \rightarrow \mathbb{C}$, whose value on $e_{i_1} \odot \dots \odot e_{i_k} \in S^k \mathbb{C}^n$ precisely equals the classically defined cumulant

$$KX_k(e_{i_1} \odot \dots \odot e_{i_k}) = c_k(x_{i_1}, \dots, x_{i_n}).$$

Now let $V \rightarrow CV \rightarrow \mathbb{C}$ be an algebraic cone on V . Because the first map is an algebra map, Proposition 3.4 applies and the cumulant of X in V equals the cumulant of $\tilde{X} : \mathbb{C}^n \rightarrow V \rightarrow CV$. Thus, the classical cumulants can be computed within the cone CV . Proposition 3.2 applies to the second map of $V \rightarrow CV \rightarrow \mathbb{C}$ so the only homotopy random variables that have the same cumulants are in fact homotopic in CV —all the possible relations among the expectations of random variables in V have been encoded in the differential in CV . Moreover, for simple collections of homotopy random variables like \tilde{X} , whose components $S^k \mathbb{C}^n \rightarrow CV$ are zero for $k > 1$, many homotopy algebra computations can be reduced to simpler homology calculations in CV .

Remark 3.8. The construction of an algebraic cone in the proof of Lemma 3.6, which says an algebraic cone exists for any probability space, should be thought of an existence argument rather than a construction. The proof uses the kernel of the expectation but in practice, one may not know much about this kernel and therefore may not have a description of the algebraic cone constructed in the proof of Lemma 3.6 that is explicit enough for calculations. Instead, one might find an algebraic cone by other means, say via a relevant group action as described in the next section, and then the homotopy-algebra tools available may be used to carry out calculations that otherwise involve transcendental methods.

4. GROUP ACTIONS

In Section 2 of [4], there is a construction of a homotopy probability space from a group acting on an ordinary probability space with a group-invariant expectation. The relations among the expectation values that arise from the group action become encoded in the differential. In the special case that *all* relations among expectation values arise from the group action, and assuming an additional vanishing condition, this construction yields an algebraic cone.

Suppose that G is a Lie group acting on an ordinary probability space V and that the expectation map e is G -equivariant. That is, $e(gx) = e(x)$ for all $g \in G$. In good cases (say G is compact and simply connected) the associated Lie algebra action $\mathfrak{g} \rightarrow \text{End}(V)$ determines the action of G . If G captures all relations among expectations: $e(x) = e(y)$ if and only if $y = gx$ for some $g \in G$, then all relations among expectations will be encoded in the \mathfrak{g} action which satisfies $e(\lambda x) = 0$ for all $\lambda \in \mathfrak{g}$.

In this situation, $C(\mathfrak{g}, V)$, the Chevalley-Eilenberg cochain complex with values in the module V , produces a homotopy probability space that with a suitable vanishing condition on higher cohomology is an algebraic cone. This cochain complex is a non-positively graded complex

$$\cdots \rightarrow \mathfrak{g}^* \otimes V \rightarrow V \rightarrow 0$$

with V in degree zero. Further, the degree zero cohomology is precisely the co-invariants $V/V_{\mathfrak{g}}$ —an element of V is in the image of the differential if and only if it has the form λx for some $\lambda \in \mathfrak{g}$. The complex $C(\mathfrak{g}, V)$ and the differential can be described explicitly as follows. Let $\{\lambda_1, \dots, \lambda_n\}$ be a basis for \mathfrak{g} and let $\rho_i : V \rightarrow V$ denote the action of λ_i on V . Let $\mathfrak{g}[-1]$ denote \mathfrak{g} with its degree shifted by 1, so that an element $\lambda \in \mathfrak{g}[-1]$ has degree 1 and an element $\eta \in (\mathfrak{g}[-1])^*$ has degree -1 .

$$C(\mathfrak{g}, V) = \text{hom}(S(\mathfrak{g}[-1]), V) \simeq S(\mathfrak{g}[-1])^* \otimes V.$$

Define a differential $d : CV \rightarrow CV$ by

$$(1) \quad d = \sum_{i=1}^n \frac{\partial}{\partial \eta_i} \otimes \rho_i + \sum_{i < j=1}^n \sum_{k=1}^n f_{ij}^k \eta_k \frac{\partial^2}{\partial \eta_i \partial \eta_j} \otimes 1$$

where $\{\eta_1, \dots, \eta_n\}$ is the basis for $\mathfrak{g}^*[1]$ dual to $\{\lambda_1, \dots, \lambda_n\}$ and the f_{ij}^k are the structure constants $[\lambda_i, \lambda_j] = \sum_{k=1}^n f_{ij}^k \lambda_k$.

Since $H^0(CV, d) = V/V_{\mathfrak{g}}$ and the expectation is invariant, extending the expectation $e : CV \rightarrow \mathbb{C}$ to be zero on $S(\mathfrak{g}[-1])^*$ is a chain map. Moreover, since it is assumed that $e(x) = 0$ if and only if $x \in V_{\mathfrak{g}}$, the expectation $CV \rightarrow \mathbb{C}$ induces an isomorphism in H^0 . If, in addition, the cohomology $H^i(CV, d)$ vanishes in all other degrees, the expectation $CV \rightarrow \mathbb{C}$ is a quasi-isomorphism. This makes $V \rightarrow CV \rightarrow \mathbb{C}$ a cone on V , in fact an algebraic cone since CV is a free extension of V .

5. AN EXPLICIT HOMOTOPY IN THE GAUSSIAN EXAMPLE

In [2], the following cone on the ordinary probability space for the one variable Gaussian was constructed. See also section 3.2 of [4], section 2.3 of [3] and references therein.

Definition 5.1. Let $V = \mathbb{C}[x, \eta]$ where x is in degree zero and η is in degree -1 (so in particular, $\eta^2 = 0$). Define the expectation as

$$e(p(x) + q(x)\eta) = \frac{\int_{-\infty}^{\infty} p(x) e^{-\frac{x^2}{2}}}{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}}$$

and the differential as

$$d(p(x) + q(x)\eta) = q'(x) - xq.$$

We call this probability space the *homotopy Gaussian*.

Remark 5.2. The one dimensional abelian Lie group \mathbb{R} acts on \mathbb{R} by translations $x \mapsto x + g$ and induces an action on the space of functions integrable with respect to the measure $e^{-\frac{x^2}{2}} dx$ by

$$f(x) \mapsto f(x + g) \exp\left(-\frac{g^2 + 2xg}{2}\right)$$

which can be seen to be integration-invariant by a simple change of variables. The induced action of the one dimensional abelian Lie algebra \mathbb{R} is generated by the single map $f(x) \mapsto f'(x) - xf(x)$. Note that the Lie algebra action can be restricted to the subspace

of polynomials. The Chevalley-Eilenberg complex with values in the space of polynomials is precisely the homotopy Gaussian. The term $\sum f_{ij}^k \eta_k \frac{\partial^2}{\partial \eta_i \partial \eta_j} \otimes 1$ in the differential in Equation (1) vanishes since here the Lie algebra is abelian.

Also, as with any abelian Lie algebra, the higher Chevalley-Eilenberg cohomology of the homotopy Gaussian vanishes making it an algebraic cone on the ordinary Gaussian (the degree zero part of the homotopy Gaussian).

For the remainder of this section, assume that X and Y are two ordinary random variables in the homotopy Gaussian; that is, $X(1) = p(x)$ and $Y(1) = q(x)$ with all higher terms zero. We will construct an explicit homotopy between X and Y in the case that they have the same cumulants $\kappa_i : S^i \mathbb{C} \rightarrow \mathbb{C}$. The reader may generalize to homotopy random variables with higher terms and to collections of homotopy random variables.

Lemma 5.3. *In the homotopy Gaussian, define*

$$y_n = -\eta \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n+1-2j} (n-1)!!}{(n+1-2j)!!}.$$

Then $dy_n = x^n - E(x^n)$.

Proof. The proof is a direct computation which is inductive. \square

Note that y_n as $n \geq 1$ varies span $\eta \mathbb{C}[x]$, so that the collection of dy_n span the image of d .

Lemma 5.4. *Assume X and Y have the same cumulants κ_i for $i \leq n$. Then $e(p^n) = e(q^n)$.*

Proof. Examine the composition $\varphi\kappa$. The hypotheses of the lemma imply that $(\varphi\kappa X)_i = (\varphi\kappa Y)_i$ for $i \leq n$. Since $\varphi\kappa = E\varphi$, this implies that $(E\varphi X)_n = (E\varphi Y)_n$. These maps take $(1 \odot \cdots \odot 1)$ to $E(p^n)$ and $E(q^n)$, respectively. \square

Lemma 5.5. *Suppose $e(r(x)) = 0$ in the homotopy Gaussian. Then $r(x)$ can be written uniquely as*

$$r(x) = \sum_{i=1}^N a_i (x^i - E(x^i))$$

where N is the degree of r .

Proof. The element $r(x)$ is closed under d . Since e is a quasi-isomorphism, $r(x)$ must be exact. \square

This motivates the following definition.

Definition 5.6. Let $e(r(x)) = 0$. Then $h(r)$ is defined as follows:

$$h(r) = \sum_{i=1}^N a_i y_i$$

so that $d(h(r)) = r(x)$.

Definition 5.7. Define an L_∞ homotopy Λ from $(\mathbb{C}, 0)$ to (V, d) as follows:

$$\Lambda_n(1 \odot \cdots \odot 1) = p^n + t(q^n - p^n) + h(p^n - q^n)dt$$

The components are evidently closed and because there are no higher brackets in (V, d) this is an L_∞ homotopy. Evaluation reveals that it is an L_∞ homotopy between φX and φY . We will shorten $\Lambda_n(1 \odot \cdots \odot 1)$ to Λ_n .

To get an L_∞ homotopy between X and Y it is now only necessary to compose with φ^{-1} .

Construction. Define the following collection of linear maps from $S^n\mathbb{C}$ to V .

$$H_n(1 \odot \cdots \odot 1) = \sum_{k=1}^n \sum_{P_k(n)} (-1)^{k-1} (k-1)! \Lambda_{p_1} \cdots \Lambda_{p_k}$$

Here $P_k(n)$ denotes partitions of n into k parts of size p_1, \dots, p_k .

Proposition 5.8. *The previous construction is an L_∞ homotopy from $(\mathbb{C}, 0)$ to (V, d^φ) between X and Y .*

Proof. Lemmas 5.3 through 5.5, along with a quick computation of φ^{-1} , yield the result. \square

Remark 5.9. If two collections of homotopy random variables are homotopic, that means that those collections are indistinguishable by cumulants. However, this does not mean that the collections will remain indistinguishable if they are extended to larger collections of homotopy random variables. For instance, in the one variable Gaussian, we can define homotopy random variables X and Y where $X(1) = x$ and $Y(1) = -x$ with all higher maps zero. These homotopy random variables are homotopic and indistinguishable.

Define collections of two homotopy random variables \overline{X} and \overline{Y} where

$$\begin{aligned} \overline{X}((1, 0)) &= x & \overline{Y}((1, 0)) &= -x \\ \overline{X}((0, 1)) &= 1 & \overline{Y}((0, 1)) &= 1 \end{aligned}$$

with all higher maps zero. Then \overline{X} and \overline{Y} are not homotopic. Indeed, the second cumulant distinguishes them.

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